Problem Set 3

Due December 2

1) a) Consider a weakly damped linear harmonic oscillator driven by white noise.

i) Derive the fluctuation spectrum at thermal equilibrium.

ii) What value of forcing is required to achieve stationarity at temperature T?

b) Now consider a forced nonlinear oscillator

$$\ddot{x} + \gamma x + \omega_0^2 x + \alpha x^3 = \tilde{f}.$$

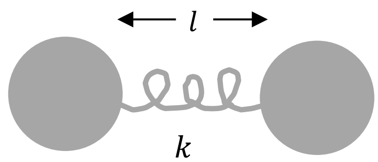
Again, assume $\tilde{f}$ is white noise. Characterize the equilibrium fluctuation spectrum. Hint: You may find it useful to review Section 29 of "Mechanics", by Landau and Lifshitz.

2) a) Derive the Fokker–Planck Equation for $f(\underline{x}, \underline{v}, t)$ for sedimentary particles in a fluid. Discuss the physics of all terms.

b) Now derive the Schmoluchowski equation for the above; solve it.

c) How might one get from $a \to b$ directly?

3) Consider an elastic dumbbell of Stokesian particles in a fluid flow $\underline{v}(\underline{x}, t)$, at temperature T.



a) Derive the Fokker–Planck equation for the length l.

b) What is the mean square length l?

Assume the dumbbell has spring constant k. The fluid has viscosity ν.

c) Now take the flow as turbulent, so $\underline{v}(\underline{x}, t) = \langle \underline{v}(\underline{x}, t) \rangle + \tilde{v}(\underline{x}, t)$, where $\tilde{v}$ is random. Repeat a) and b), above.

4) Give a general, but purely classical, derivation of the Fluctuation–Dissipation Theorem.

5) Prove the Fluctuation–Dissipation Theorem for a multi-field system. (You may find Landau and Lifshitz's "Statistical Physics" useful, here.)

6) a) Derive the Fokker–Planck equation for $\langle f(\underline{p}, t)\rangle$, the mean distribution function for a system for particles moving according to the Hamiltonian equations of motion for Hamiltonian $H = H(\underline{p}, q)$. Assume the average is over $\underline{q}$.

b) Show that in the F–P equation, the drift and diffusion partially cancel, so the F–P equation simplifies to

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial D_p}{\partial p}\frac{\partial\langle f\rangle}{\partial p}$$

c) What is D_p?

7) a) State and prove the Central Limit Theorem starting from the Chapman–Kolmogorov equation. Chandrasekhar is a good reference.

b) Give a concise summary of when the Central Limit Theorem applies — i.e. what conditions must be met?

c) What happens if the probability of step size x is:

$$p(x) = {}^{1}\!/_{(1+x^4)}?$$

8) Consider a function q which satisfies:

$$\tau \frac{\partial q}{\partial t} = -a(T, T_c)q - bq^3 + \tilde{f}$$

Here $\langle \tilde{f}^2 \rangle = \left|\tilde{f}_0\right|^2 \tau_c \delta(t_1 - t_2)$.

a) Derive the Fokker–Planck equation for $P(q,t)$. Solve and discuss the stationary solution for $T > T_c$, $T < T_c$, $T \neq T_c$.

b) How does $P(q,t)$ evolve if T passes adiabatically thru T_c? Here "adiabatically" means $\tau_c \left(\frac{\partial T}{\partial t}\right) T \ll 1$.

c) Discuss the behavior when

$$a = a_0 + \tilde{a}$$

$$\langle \tilde{a}^2 \rangle = \bar{a}^2 \tau_0 \delta(t_1 - t_2)$$

9) a) Generalize the calculation of the current between stable states A, B in the Kramers problem to the case where $n_B \neq 0$.

b) For what ratio n_A/n_B does $j = 0$?

c) Calculate j for n_B finite and $n_A \to 0$. Compare this to the value of j calculated in class.